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Two-component KP hierarchy and the classical Boussinesq equation

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Abstract. It has been shown that solutions to the KP equation and other equations in the KP hierarchy take a Wronskian form. We extend these ideas to two-component Wronskians and the Hirota equations which they satisfy. By a reduction procedure we show how dependence on some of the variables can be removed and look in detail at the classical Boussinesq equation and its modifications and show how these equations fit into the two-component KP hierarchy.

1. Introduction

Hirota's method of solving nonlinear evolution equations has proved very successful in producing N-soliton solutions. One of the most widely studied equations is the Kadomtsev-Petviashvili (KP) equation [1]

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{1.1}$$

Using a transformation of variables

$$u = 2\partial_x^2[\ln f]$$

this equation can be expressed in Hirota bilinear form and an N-soliton solution can be found. In the early 1980s work was done on categorizing the Hirota equations in the KP hierarchy and associated hierarchies [2-4]. The solutions to the equations are written down as the so-called 'τ-functions'. In the paper of Jimbo and Miwa [2] the two-component KP hierarchy is discussed. Here we shall set up a formalism in which τ-functions for the two-component KP hierarchy are expressed as 'two-component Wronskians' and Hirota equations of this hierarchy—of which a partial list appears in [2]—are obtained as vanishing determinants analogous to the Plücker relation of the single-component hierarchy [3-5].

We pay particular attention to the classical Boussinesq equation and its modifications. A derivation using the theory of water waves has been given by Whitham [6]. Using a suitable transformation of variables, this equation can be written in bilinear form and by using a rather unusual limiting procedure called the '$pq = 0$' reduction; Hirota could relate the classical Boussinesq equation to equations of the KP and modified KP hierarchies [7, 8]. A more direct route for obtaining solutions of this equation can be made by fitting it into the two-component hierarchy of Jimbo and Miwa.

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In addition to the Boussinesq equation, there exists a first modification, obtained from the original equation by a Miura transformation and also a second modification obtained from a further Miura transformation [9, 10]. By careful choice of transformation we have been able to show that these systems are also part of the two-component KP hierarchy.

2. Wronskian and two-component Wronskian determinants

2.1. Wronskians

We will discuss the properties of certain Wronskian determinants and the Hirota equations that they satisfy. We have stated these properties in a form that will enable us to show how some familiar ideas may be extended in a straightforward way to the two-component case. Let $x=(x_1, x_2, \ldots)$ be a sequence of independent variables and, for $i=1, \ldots, N$, let $\varphi_i(x)$, be a set of functions depending on the sequence x and satisfying the linear partial differential equations

$$\partial_j \varphi_i = \partial^j \varphi_i \tag{2.1}$$

where ∂_j is the partial x_j-derivative for $j \in \mathbb{N}$ and $\partial = \partial_1$. For $p \in \mathbb{Z}$, we denote by $\boldsymbol{\varphi}^p(x)$ the pth x_1-derivative of the column vector $(\varphi_1, \varphi_2, \ldots, \varphi_N)^{\mathrm{T}}$. If $p<0$ then $\boldsymbol{\varphi}^p(x)$ has only a formal interpretation as a vector whose $(-p)$th x_1-derivative is $\boldsymbol{\varphi}(x)$.

Now consider the $N \times (N+k)$ matrix formed from $N+k$ column vectors $\boldsymbol{\varphi}^p(x)$:

$$M_s^k(x) = (\boldsymbol{\varphi}^s, \boldsymbol{\varphi}^{s+1}, \ldots, \boldsymbol{\varphi}^{s+N+k-1}). \tag{2.2}$$

If $k=0$ then we drop the superscript on $M_s^k(x)$ and we may define the τ-function

$$\tau_s(x) = \det(M_s(x)). \tag{2.3}$$

This is a Wronskian determinant of the N functions φ_i. Notice that this definition is such that by taking $\partial^t \varphi_i$ $(i=1, \ldots, N)$ in (2.1) the labelling is shifted so that τ_s becomes τ_{s+t}. The effect of this is that the precise labelling is not crucial, it is the *relative* labelling of the τ-functions appearing in a particular expression that is important. This fact is used several times below.

Now let $x' = (x'_1, x'_2, \ldots)$ be a second sequence of independent variables. When $k_1 + k_2 = 0$ the determinant

$$\Delta_{s_1,s_2}^{k_1,k_2}(x, x') = \det \begin{bmatrix} M_{s_1}^{k_1}(x) & 0 \\ 0 & M_{s_2}^{k_2}(x') \end{bmatrix} \tag{2.4}$$

is well defined, and without loss of generality we assume that $k_1 \leqslant k_2$. If $k_1 = k_2$ $(=0)$ then

$$\Delta_{s_1,s_2}^{0,0}(x, x') = \tau_{s_1}(x)\tau_{s_2}(x') \tag{2.5}$$

and otherwise, for $k>0$,

$$\Delta_{s_1,s_2}^{-k,k}(x, x') = 0. \tag{2.6}$$

It may be shown that provided $s_1 - s_2 \leqslant 1$ and $k>0$, the expansion of

$$\Delta_{s_1,s_2}^{-k,k}(x, x) = \det \begin{bmatrix} M_{s_1}^{-k}(x) & M_{s_2}^{k}(x) \\ 0 & M_{s_2}^{k}(x) \end{bmatrix} = 0 \tag{2.7}$$

by its $N\times N$ minors may be written in the form

$$H(\mathrm{D}_1, D_2, \ldots)\tau_{s_1}(\boldsymbol{x})\cdot\tau_{s_2}(\boldsymbol{x}) \equiv H(\partial_1-\partial_1', \partial_2-\partial_2', \ldots)\tau_{s_1}(\boldsymbol{x})\tau_{s_2}(\boldsymbol{x}')|_{\boldsymbol{x}'=\boldsymbol{x}}=0 \tag{2.8}$$

where H is a polynomial. That is, it is a Hirota equation [11] that is satisfied by the Wronskian determinants (2.3). This is but a special case of a result [12] which is most conveniently expressed in terms of a generalization of the Hirota derivatives D_i.

Define operators $\tilde{\mathrm{D}}_i$ for $i\in\mathbb{N}$ by

$$P(\tilde{\mathrm{D}}_1, \tilde{\mathrm{D}}_2, \ldots)\Delta^{-k,k}_{s_1,s_2}(\boldsymbol{x}, \boldsymbol{x}) \equiv P(\partial_1-\partial_1', \partial_2-\partial_2', \ldots)\Delta^{-k,k}_{s_1,s_2}(\boldsymbol{x}, \boldsymbol{x}')|_{\boldsymbol{x}'=\boldsymbol{x}}=0 \tag{2.9}$$

P a polynomial, cf (2.8). By virtue of (2.5), this definition coincides precisely with that of Hirota derivatives, (2.8), when $k=0$. We have that for any polynomial P, ($s_1-s_2\leqslant 1$ and $k>0$),

$$P(\tilde{\mathrm{D}}_1, \tilde{\mathrm{D}}_2, \ldots)\Delta^{-k,k}_{s_1,s_2}(\boldsymbol{x}, \boldsymbol{x}) = H(\mathrm{D}_1, \mathrm{D}_2, \ldots)\tau_{s_1}(\boldsymbol{x})\cdot\tau_{s_2}(\boldsymbol{x})=0. \tag{2.10}$$

The left-hand side of (2.10) vanishes because it is a linear combination of determinants similar to (2.7). (We note that it seems that the restriction on s_1 and s_2 may be removed if one introduces a second sequence of 'negative weight' variables $\boldsymbol{y}$ such that

$$\frac{\partial\varphi_i}{\partial y_j}=\partial^{-j}\varphi_i$$

but this issue will not be pursued further here.)

As a familiar example, we have the Hirota form of the KP equation

$$\Delta^{-2,2}_{0,0}(\boldsymbol{x}, \boldsymbol{x}) = \pm\tfrac{1}{12}(\mathrm{D}_1^4+3\mathrm{D}_2^2-4\mathrm{D}_3\mathrm{D}_1)\tau_0\cdot\tau_0=0 \tag{2.11}$$

with $x_1=x$, $x_2=y$ and $x_3=-4t$ [13, 14] and

$$\Delta^{-1,1}_{0,1}(\boldsymbol{x}, \boldsymbol{x}) = \pm\tfrac{1}{2}(\mathrm{D}_1^2+\mathrm{D}_2)\tau_0\cdot\tau_1=0 \tag{2.12}$$

$$\tilde{\mathrm{D}}_{x_1}\Delta^{-1,1}_{0,0}(\boldsymbol{x}, \boldsymbol{x}) = \pm\tfrac{1}{6}(\mathrm{D}_1^3-3\mathrm{D}_2\mathrm{D}_1-2\mathrm{D}_3)\tau_0\cdot\tau_1=0 \tag{2.13}$$

which together constitute the Hirota form of a modified KP equation. The sign in each of the above equations is determined by the parity of N and, since the right-hand side is zero in each case, is of no significance. The Hirota equations generated in this way make up the KP and modified KP hierarchies [2, 4].

Often one wishes to consider τ-function which depend only on a subsequence of $\boldsymbol{x}$. Such τ-functions do not in general satisfy the Hirota equations obtained by omitting the Hirota derivatives corresponding to the variables not present in this subsequence. Rather, this reduction process is carried out by making an appropriate choice of the functions φ_i satisfying (2.1) which result in the τ-function being dependent only on the required subsequence. Here we will only be concerned with the case where we wish to have τ-functions dependent only on odd-index variables. This corresponds to the 'A_∞ to $A_1^{(1)}$' reduction described in Jimbo and Miwa [2]. We shall describe how this reduction may be achieved for soliton-type solutions.

For $i=1,\ldots,N$ let

$$\varphi_i(x)=\alpha_i\exp(\xi(\boldsymbol{x}, p_i))+\beta_i\exp(\xi(\boldsymbol{x}, q_i)) \tag{2.14}$$

where

$$\xi(\boldsymbol{x}, k)=\sum_{j=1}^{\infty} k^j x_j \tag{2.15}$$

and α_i, β_i, p_i and q_i are arbitrary constants. Clearly this choice of the φ_i is compatible with (2.1). For each $i=1,\ldots,N$ take $q_i=-p_i$ and so

$$\varphi_i(\boldsymbol{x})=\exp(\xi(\boldsymbol{x}_{\mathrm{e}},p_i))[\alpha_i\exp(\xi(\boldsymbol{x}_0,p_i))+\beta_i\exp(-\xi(\boldsymbol{x}_0,p_i))]$$

where $\boldsymbol{x}_{\mathrm{e}}$ (respectively $\boldsymbol{x}_{\mathrm{o}}$) is obtained from $\boldsymbol{x}$ by omitting x_j for each odd (respectively even) j. As a result of this we have

$$\tau_s(\boldsymbol{x})=\pi(\boldsymbol{x}_{\mathrm{e}})\tau_s(\boldsymbol{x}_o) \tag{2.16}$$

where

$$\pi(\boldsymbol{x}_{\mathrm{e}})=\prod_{i=1}^{N}\exp(\xi(\boldsymbol{x}_{\mathrm{e}},p_i)) \tag{2.17}$$

does not depend on the subscript s. Any Hirota equation (2.8) may be written as

$$\left(\sum_{k=1}^{M}H_k^o(\mathrm{D}_{x_1},\mathrm{D}_{x_3},\ldots)H_k^{\mathrm{e}}(\mathrm{D}_{x_2},\mathrm{D}_{x_4},\ldots)\right)\tau_{s_1}\cdot\tau_{s_2}=0 \tag{2.18}$$

where H_k^o and H_k^{e} are polynomials in the odd and even Hirota derivatives, respectively, and M is the number of terms in the polynomial. For the reduced τ-function (2.16) we have

$$\sum_{k=1}^{M}(H_k^o(\mathrm{D}_1,\mathrm{D}_3,\ldots)\tau_{s_1}(\boldsymbol{x}_o)\cdot\tau_{s_2}(\boldsymbol{x}_o))(H_k^{\mathrm{e}}(\mathrm{D}_2,\mathrm{D}_4,\ldots)\pi(\boldsymbol{x}_{\mathrm{e}})\cdot\pi(\boldsymbol{x}_{\mathrm{e}}))=0 \tag{2.19}$$

and because of the particular form of $\pi(\boldsymbol{x}_{\mathrm{e}})$, whenever the degree of $H_k^{\mathrm{e}}>0$

$$H_k^{\mathrm{e}}(\mathrm{D}_2,\mathrm{D}_4,\ldots)\pi(\boldsymbol{x}_{\mathrm{e}})\cdot\pi(\boldsymbol{x}_{\mathrm{e}})=0. \tag{2.20}$$

Hence the Hirota equation (2.8) for the reduced τ-functions (2.14) is

$$H(\mathrm{D}_1,0,\mathrm{D}_3,0,\mathrm{D}_5,\ldots)\tau_{s_1}(\boldsymbol{x}_o)\cdot\tau_{s_2}(\boldsymbol{x}_o)=0. \tag{2.21}$$

The reduction is thus achieved by choosing the φ_i as in (2.14), replacing $\tau(\boldsymbol{x})$ with $\tau_s(\boldsymbol{x}_o)$ and D_{2k} with 0. Notice that these reduced τ-functions have the property

$$\tau_{2s}(\boldsymbol{x}_o)=\left(\prod_{i=1}^{N}p_i\right)^s\tau_0(\boldsymbol{x}_o)\qquad\text{and}\qquad\tau_{2s+1}(\boldsymbol{x}_o)=\left(\prod_{i=1}^{N}p_i\right)^s\tau_1(\boldsymbol{x}_o) \tag{2.22}$$

so that there are just two essentially different reduced τ-functions $\tau_0(\boldsymbol{x}_o)$ and $\tau_1(\boldsymbol{x}_o)$ in the sense that any Hirota equation involving reduced τ-functions τ_s and τ_t may always be rewritten in terms of τ_0 or τ_0 and τ_1, depending on the parity of s and t.

2.2. Two-component Wronskians

The above notions will now be extended to determinants containing two blocks of columns depending on different sets of variables. Each block will consist of columns obtained by differentiating its first column and for this reason we call such determinants *two-component Wronskians.*

For $j=1,2$, let $\boldsymbol{x}^{(j)}=(x_1^{(j)},x_2^{(j)},\ldots)$ be sequences of independent variables and, for $1\leqslant i\leqslant N^{(1)}+N^{(2)}=N$, let $\varphi_i^{(j)}(\boldsymbol{x}^{(j)})$ be sets of functions satisfying

$$\partial_k^{(j)}\varphi_i^{(j)}=\partial^{(j)^k}\varphi_i^{(j)} \tag{2.23}$$

where $\partial_k^{(j)}$ denotes the $x_k^{(j)}$ derivative and $\partial^{(j)}=\partial_1^{(j)}$ (cf (2.1)). In analogy with the single-component case, $\varphi^{(j)k}$ is the kth $x_1^{(j)}$ derivative of the column vector $(\varphi_1^{(j)},\ldots,\varphi_N^{(j)})^{\mathrm{T}}$ and define the $N\times(N+k+1)$ matrix

$$M_{s;t}^{k;l}(\boldsymbol{x}^{(1)};\boldsymbol{x}^{(2)})=(\boldsymbol{\varphi}^{(1)^s},\ldots,\boldsymbol{\varphi}^{(1)^{s+N^{(1)}+k-1}};\boldsymbol{\varphi}^{(2)^t},\ldots,\boldsymbol{\varphi}^{(2)^{t+N^{(2)}+l-1}}) \quad (2.24)$$

For $l=-k$ we may define the *two-component τ-function*

$$\tau_{s;t}^{k;-k}=\det(M_{s;t}^{k;-k}) \quad (2.25)$$

which is a two-component Wronskian determinant. Here and throughout we use the semi-colon to separate the subscripts and superscripts etc to indicate parts referring to the different components.

The determinants,

$$\Delta_{s_1,s_2;t_1,t_2}^{k_1,k_2;l_1,l_2}(\boldsymbol{x}^{(1)},\boldsymbol{x}'^{(1)};\boldsymbol{x}^{(2)},\boldsymbol{x}'^{(2)}) =\det\begin{bmatrix} M_{s_1;t_1}^{k_1;l_1}(\boldsymbol{x}^{(1)};\boldsymbol{x}^{(2)}) & 0 \\ 0 & M_{s_2;t_2}^{k_2;l_2}(\boldsymbol{x}'^{(1)};\boldsymbol{x}'^{(2)})\end{bmatrix} \quad (2.26)$$

may be defined if $k_1+l_1+k_2+l_2=0$ where we take $k_1+l_1\leqslant k_2+l_2$. Furthermore, determinants such as (2.26) give rise to Hirota equations amongst the τ-functions $\tau_{s;t}^{k;-k}$ in much the same way as described for the single-component case. For $k_1+l_1=k_2+l_2$ the left-hand side of (2.26) is a product of τ-functions, and if $k_1+l_1<k_2+l_2$, $s_1-s_2\leqslant 1$ and $t_1-t_2\leqslant 1$, for any polynomial P

$$P(\tilde{\mathrm{D}}_1^{(1)},\tilde{\mathrm{D}}_2^{(1)},\ldots;\tilde{\mathrm{D}}_1^{(2)},\tilde{\mathrm{D}}_2^{(2)},\ldots)\Delta_{s_1,s_2;t_1,t_2}^{k_1,k_2;l_1,l_2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(1)};\boldsymbol{x}^{(2)},\boldsymbol{x}^{(2)})=0 \quad (2.27)$$

is a Hirota equation involving the τ-functions (2.25). The operators $\tilde{\mathrm{D}}_i^{(j)}$, are an obvious two-component extension of the generalized Hirota derivatives defined in (2.9). The big difference between this case and the single-component case is the fact that, whereas (2.10) gives a Hirota equation involving at most two τ-functions, its analogue (2.27) may give one containing any number of different τ-functions. This inevitably leads to a much richer structure and wider variety of Hirota equations.

The soliton system which plays the fundamental role, analogous to that which the KP equation plays for single-component Wronskians, is the Davey-Stewartson equations. Consider the following examples of (2.27):

$$\Delta_{0,0;0,0}^{-1,2;-1,0}=\pm[\mathrm{D}_2^{(1)}+\mathrm{D}_1^{(1)^2}]\tau_{0,0}^{0,0}\cdot\tau_{0,0}^{1,-1}=0 \quad (2.28a)$$

$$\Delta_{0,0;0,0}^{-2,1;1,0}=\pm[\mathrm{D}_2^{(1)}-\mathrm{D}_1^{(1)^2}]\tau_{0,0}^{0,0}\cdot\tau_{0,0}^{-1,1}=0 \quad (2.28b)$$

$$\Delta_{0,0;0,0}^{-1,1;-1,1}=\pm(\mathrm{D}_1^{(1)}\mathrm{D}_1^{(2)}\tau_{0,0}^{0,0}\cdot\tau_{0,0}^{0,0}-2\tau_{0,0}^{1,-1}\cdot\tau_{0,0}^{-1,1})=0 \quad (2.28c)$$

together with those equations obtained by interchanging the components in (2.28*a*, *b*). Now perform the invertible change of independent variables,

$$x_1^{(1)}=x+\mathrm{i}y \qquad x_1^{(2)}=x-\mathrm{i}y \qquad x_2^{(1)}=2\mathrm{i}t \qquad x_2^{(2)}=-2\mathrm{i}t \quad (2.29)$$

let $N^{(2)}=N^{(1)}$ and choose $\varphi_i^{(2)}$ to be the complex conjugate of $\varphi_i^{(1)}$ for $i=1,\ldots,2N$ so that $\tau_{0,0}^{0,0}\equiv F(x,y,t)$ is real and $\tau_{0,0}^{1,-1}\equiv G(x,y,t)$ and $\tau_{0,0}^{-1,1}\equiv G^*(x,y,t)$ are complex conjugate. The Hirota equations (2.28) may then be written as

$$(\mathrm{iD}_t+\mathrm{D}_x^2-\mathrm{D}_y^2)G\cdot F=0 \quad (2.30a)$$

$$(\mathrm{D}_x^2+\mathrm{D}_y^2)F\cdot F=8GG^* \quad (2.30b)$$

which constitute the Hirota form of the Davey–Stewartson equations [15],

$$\mathrm{i}u_t + u_{xx} - u_{yy} + 4u\Sigma - 8u|u|^2 = 0 \tag{2.31a}$$

$$\Sigma_{xx} + \Sigma_{yy} = 4(|u|^2)_{xx}. \tag{2.31b}$$

The process of reduction for two-component Wronskians is very similar to the single-component case. The τ-functions that one obtains after this reduction depend on just one sequence of variables rather than one for each component. To achieve this we again perform a non-singular change of variables reminiscent of (2.29), for $k \in \mathbb{N}$,

$$x_k^{(1)} = x_k + y_k \qquad \text{and} \qquad x_k^{(2)} = (-1)^{k+1}(x_k - y_k) \tag{2.32}$$

so that

$$\mathrm{D}_{x_k}^{(1)} = \tfrac{1}{2}(\mathrm{D}_{x_k} + \mathrm{D}_{y_k}) \qquad \text{and} \qquad \mathrm{D}_{x_k}^{(2)} = \tfrac{1}{2}(-1)^{k+1}(\mathrm{D}_{x_k} - \mathrm{D}_{y_k}) \tag{2.33}$$

and thereby the τ-functions become dependent on the two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$. For soliton-type solutions one takes

$$\varphi_i^{(1)} = \alpha_i \exp(\xi(\boldsymbol{x}^{(1)}, p_i) \qquad \text{and} \qquad \varphi_i^{(2)} = \beta_i \exp(\xi(\boldsymbol{x}^{(2)}, q_i)) \tag{2.34}$$

$(i = 1, \ldots, N^{(1)} + N^{(2)}$ and if one chooses $q_i = -p_i$ then

$$\varphi_i^{(1)} = \alpha_i \exp(\xi(\boldsymbol{x}, p_i)) \exp(\xi(\boldsymbol{y}, p_i))$$

and

$$\varphi_i^{(2)} = \beta_i \exp(-\xi(\boldsymbol{x}, p_i) \exp(\xi(\boldsymbol{y}, p_i))$$

and so

$$\tau_{s;t}^{k;-k}(\boldsymbol{x}^{(1)}; \boldsymbol{x}^{(2)}) = \left(\prod_{i=1}^{N^{(1)}+N^{(2)}} \exp(\xi(\boldsymbol{y}, p_i))\right) \tau_{s;t}^{k;-k}(\boldsymbol{x}; \boldsymbol{x}). \tag{2.37}$$

As in the single-component case, the $\boldsymbol{y}$ dependence is effectively eliminated because it only appears in an exponential factor. Also, the reduced τ-functions are related by

$$\tau_{s;t}^{k;-k}(\boldsymbol{x}; \boldsymbol{x}) = \left((-1)^{N^{(2)}+k} \prod_{i=1}^{N^{(1)}+N^{(2)}} p_i\right)^j \tau_{s-j;t-j}^{k;-k}(\boldsymbol{x}; \boldsymbol{x}). \tag{2.38}$$

The reduction is thus achieved by replacing both $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ with $\boldsymbol{x}$ in the τ-functions, and replacing $\mathrm{D}_{x_k}^{(1)}$ with $\frac{1}{2}\mathrm{D}_{x_k}$ and $\mathrm{D}_{x_k}^{(2)}$ with $\frac{1}{2}(-1)^{k+1}\mathrm{D}_{x_k}$ in the equations they satisfy.

3. The classical Boussinesq equation and its modifications

The classical Boussinesq equation [7, 8] may be written in the form

$$u_t = \{(1+u)v\}_x + v_{xxx} \qquad v_t = (u + v^2/2)_x. \tag{3.1}$$

By introducing the variables τ and τ'

$$u = -1 + 2(\ln \tau\tau')_{xx} \qquad v = 2(\ln \tau'/\tau)_x \tag{3.2}$$

the equations (3.1) can be written in terms of Hirota derivatives as

$$(\mathrm{D}_x^2 + \mathrm{D}_t)\tau \cdot \tau' = 0 \qquad (\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^3)\tau \cdot \tau' = 0. \tag{3.3}$$

In addition to the classical Boussinesq equations there are associated systems of equations also describing dispersive water waves [10]. These we refer to as modified forms of the classical Boussinesq equations.

The first modified form of the classical Boussinesq system is given by [8, 9]

$$\bar{u}_t = (\bar{u}_x - \bar{u}^2 + 2\bar{u}\bar{v})_x \qquad \bar{v}_t = -(\bar{v}_x - \bar{v}^2 + 2\bar{v}\bar{u})_x \tag{3.4}$$

which are related to (3.1) by a Miura transformation

$$v = 2(\bar{v} - \bar{u}) \qquad (1+u) = -2(\bar{u}_x + \bar{v}_x + 2\bar{u}\bar{v}). \tag{3.5}$$

The second modified form [9] is

$$\bar{\bar{u}}_t = -\tfrac{1}{2}(\bar{\bar{v}}_{xx} + \bar{\bar{v}}\bar{\bar{u}}_x - \bar{\bar{u}}^2\bar{\bar{v}})_x \tag{3.6a}$$

$$\bar{\bar{v}}_t = \tfrac{1}{2}(\bar{\bar{v}}\bar{\bar{v}}_x + \bar{\bar{u}}\bar{\bar{v}}^2 - 4\bar{\bar{u}})_x. \tag{3.6b}$$

Again these are related to the previous system by a Miura transformation

$$\bar{u} = \tfrac{1}{2}\bar{\bar{u}} - \tfrac{1}{4}(\bar{\bar{v}}_x + \bar{\bar{v}}\bar{\bar{u}}) \tag{3.7a}$$

$$\bar{v} = \tfrac{1}{2}\bar{\bar{u}} + \tfrac{1}{4}(\bar{\bar{v}}_x + \bar{\bar{v}}\bar{\bar{u}}). \tag{3.7b}$$

We shall show how each of these systems is reduced to Hirota form and also how the Miura transformations are satisfied.

The dependent variables required turn out to be two-component Wronskians, and to this end shall introduce the following notation:

$$\begin{array}{ll} F = \tau_{0;0}^{0;0} & F^{(1)} = \tau_{1;1}^{0;0} \\ G = \tau_{0;0}^{1;-1} & G^{(1)} = \tau_{1;1}^{1;-1} \\ G^* = \tau_{0;0}^{-1;1} & G^{*(1)} = \tau_{1;1}^{-1;1} \\ f = \tau_{0;1}^{1;-1} & f^* = \tau_{1;0}^{-1;1} \\ g = \tau_{1;0}^{0;0} & g^* = \tau_{0;1}^{0;0} \end{array} \tag{3.8}$$

with

$$A^{\pm} = f \pm \varepsilon g \qquad A^{\pm *} = \varepsilon f^* \pm g^* \tag{3.9}$$

where $\varepsilon = \pm 1$. As might be expected, this number of τ-functions satisfy numerous Hirota equations, even if we restrict ourselves to a low order. In appendix 1 we have listed the equations we require and also how they arise from (2.27). Using the reduction procedure in section 2, these equations reduce to

$$A^+A^{+*} + A^-A^{-*} = 2FF^{(1)} \tag{3.10a}$$

$$D_xF \cdot A^{\pm} = \mp 2GA^{\mp *} \tag{3.10b}$$

$$D_xF \cdot A^{\pm *} = \mp 2G^*A^{\mp} \tag{3.10c}$$

$$D_xA^+ \cdot A^- = 4G^{(1)}F \tag{3.10d}$$

$$D_xA^{+*} \cdot A^{-*} = 4G^{(1)*}F \tag{3.10e}$$

$$D_x^2F \cdot F = 8GG^* \tag{3.10f}$$

$$(D_t + D_x^2)F \cdot A^{\pm} = 0 \tag{3.10g}$$

$$(D_t - D_x^2)F \cdot A^{\pm *} = 0 \tag{3.10h}$$

$$(D_t - D_x^2)A^+ \cdot A^{-*} = 0 \tag{3.10i}$$

$$D_x(D_t - D_x^2)A^+ \cdot A^{-*} = 0 \tag{3.10j}$$

where $x = x_1$ and $t = x_2/2$.

Starting with the most complicated system (3.6) we shall carry out a change of variables

$$\bar{u}=\frac{\partial}{\partial x}\ln\left(\frac{FF^{(1)}}{A^+A^{-*}}\right) \tag{3.11}$$

$$\bar{v}=\frac{A^+A^{+*}-A^-A^{-*}}{FF^{(1)}} \tag{3.12}$$

where in (3.11) we could have used $\partial_x^2[\ln(F^2/A^+A^{-*})]$ since F and $F^{(1)}$ are proportional by virtue of (2.38), but we write $\bar{u}$ in the way indicated to achieve symmetry with (3.12).

To achieve the bilinearization we utilize a subsidiary result which will be used again in considering the Miura transformation (see appendix 2):

$$\bar{v}_x+\bar{u}\bar{v}=2\partial_x[\ln(A^+/A^{-*})]. \tag{3.13}$$

Now let us consider equation (3.6*a*), using equation (3.13) and (3.10*f*, *g*) we can show that (3.6*a*) is satisfied, i.e.

$$\begin{aligned}\bar{u}_t&=-\partial_x\left[\frac{\mathrm{D}_t(F\cdot A^+)}{FA^+}+\frac{\mathrm{D}_t(F\cdot A^{-*})}{F\cdot A^{-*}}\right]\\&=-\tfrac{1}{2}(\bar{v}_{xx}+\bar{u}_x\bar{v}-\bar{u}^2\bar{v})_x\end{aligned}$$

as required. Using (3.13) again and some of the other equations of (3.10) the bilinearization of (3.66) can also be achieved. By noting the Hirota equations used in the bilinearization process we obtain the Hirota form of (3.6)

$$\mathrm{D}_xF\cdot A^{\pm}=\mp 2GA^{\mp *}\qquad \mathrm{D}_xF\cdot A^{\pm *}=\mp 2G^*A^{\mp}\qquad \mathrm{D}_xA^+\cdot A^-=4G^{(1)}F$$

$$\mathrm{D}_xA^{+*}\cdot A^{-*}=4G^{(1)*}F\qquad A^+A^{+*}+A^-A^{-*}=2FF^{(1)}$$

$$(\mathrm{D}_t+\mathrm{D}_x^2)F\cdot A^{\pm}=0\qquad (\mathrm{D}_t-\mathrm{D}_x^2)F\cdot A^{\pm *}=0.$$

To obtain the change of variables appropriate to the bilinearization of (3.4), we use the Miura transformation (3.7). Using equation (3.13), equation (3.7*a*) gives

$$\bar{u}=\partial_x[\ln(F/A^+)]$$

and (3.7*b*) gives

$$\bar{v}=\partial_x[\ln(F/A^{-*})].$$

Using a similar process to the bilinearization of (3.6) described above, we obtain the Hirota form of (3.4)

$$(\mathrm{D}_t+\mathrm{D}_x^2)F\cdot A^+=0\qquad (\mathrm{D}_t-\mathrm{D}_x^2)F\cdot A^{-*}=0$$

$$\mathrm{D}_xF\cdot A^+=-2GA^{-*}\qquad \mathrm{D}_xF\cdot A^{-*}=2G^*A^-\qquad \mathrm{D}_x^2F\cdot F=8GG^*.$$

The Muira transformation (3.5) from system (3.4) to (3.1) gives

$$u+1=2(\ln A^+A^{-*})_{xx}\qquad v=2(\ln A^+/A^{-*})_x.$$

This is precisely the same form as (3.2) with τ replaced by A^{-*} and τ' replaced by A^+. Thus the Hirota form is given by

$$(\mathrm{D}_x^2-\mathrm{D}_t)A^+\cdot A^{-*}=0\qquad \mathrm{D}_x(\mathrm{D}_x^2-\mathrm{D}_t)A^+\cdot A^{-*}=0.$$

Hence we have shown that the three systems (3.1), (3.4) and (3.6) are solved by two-component (reduced) Wronskians. In the following section we compare this with the approach of Hirota to the classical Boussinesq equation (3.1) via a '$pq=0$' reduction of the single-component KP hierarchy.

4. The classical Boussinesq equation

As we have seen in section 3, the Boussinesq equations can be written in terms of Hirota bilinear forms:

$$(\mathrm{D}_x^2+\mathrm{D}_t)\tau\cdot\tau'=0 \qquad (\mathrm{D}_x\mathrm{D}_t+\mathrm{D}_x^3)\tau\cdot\tau'=0. \tag{4.1}$$

In Hirota's paper [7] these equations are related to the class of equations discussed by Jimbo and Miwa [2] and referred to as the modified KP hierarchy.

The first two equations of this hierarchy are

$$(\mathrm{D}_x^2+\mathrm{D}_t)\tau\cdot\tau'=0 \tag{4.3}$$

$$(\mathrm{D}_x^3-4\mathrm{D}_{t'}-3\mathrm{D}_x\mathrm{D}_t)\tau\cdot\tau'=0. \tag{4.4}$$

Hirota [7, 8] has shown that, by using the '$pq=C$' reduction (i.e., assuming p_jq_j is independent of j or $q_j=C/p_j$), (4.3) and (4.4) become

$$(\mathrm{D}_1^2+\mathrm{D}_2)\tau\cdot\tau'=0 \tag{4.5}$$

$$(\mathrm{D}_1^3+\mathrm{D}_1\mathrm{D}_2+4C\mathrm{D}_1)\tau\cdot\tau'=0 \tag{4.6}$$

$$(\mathrm{D}_3+\mathrm{D}_1\mathrm{D}_2+C\mathrm{D}_1)\tau\cdot\tau'=0 \tag{4.7}$$

where $x=x_1$, $t=x_2$ and $t'=x_3$.
In the limit $C\to 0$ (4.5) and (4.6) reduce to the classical Boussinesq equations.

The n-soliton solutions to equations (4.3) and (4.4) can be expressed in terms of Wronskian determinants [16, 17].

$$\tau=\tau_0^0(\boldsymbol{x}) \qquad \tau'=\tau_1^0(\boldsymbol{x}) \tag{4.8}$$

using the notation of section 2. The functions in the determinants are given by (2.14).

It may be shown that

$$(\mathrm{D}_1^3+\mathrm{D}_1\mathrm{D}_2+4C\mathrm{D}_1)\tau\cdot\tau'=4(\mathrm{D}_3+\mathrm{D}_1\mathrm{D}_2+C\mathrm{D}_1)\tau\cdot\tau'$$
$$=4(-1)^n\left[\begin{vmatrix} M_0^{-1} & \vdots & M_2^1 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & M_1^{-3}\end{vmatrix}+C\begin{vmatrix} M_0^{-1} & \vdots & M_0^1 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & M_1^{-1}\end{vmatrix}\right]. \tag{4.9}$$

Using the reduction $p_iq_i=C$ in the form

$$\left(p_i+\frac{C}{p_i}\right)\varphi_i^{(j)}=\varphi_i^{(j+1)}+C\varphi_i^{(j-1)}$$

the determinantal expression on the right-hand side of equation (4.9) can be shown to be zero after row and column manipulations.

Note that to relate with earlier work on the Wronskian method, the sign of p_i must be changed.

If we choose $\alpha_i=(p_i)^{-r}$ and $\beta_i=(q_i)^{-r}$ then this has the effect of shifting all the derivatives r places down so that τ and τ' now take the form

$$\tau=\tau_{-r}^0 \qquad \tau'=\tau_{-r+1}^0. \tag{4.10}$$

We wish to examine τ and τ' in the limit $C\to 0$. To do this we write $q_i=C/p_i$ and note that the dominant behaviour in φ_i comes from the inverse powers of the q_i when

$j<0$, whereas for $j \geqslant 0$ it is the terms involving p_i which contribute. Hence for C small

$$\tau \approx C^{-r(r+1)/2} \det[M_1 \vdots M_2] \qquad \tau' \approx C^{-r(r-1)/2} \det[M_3 \vdots M_4]$$

where

$$\begin{aligned}
(M_1)_{ij} &= p_i^{r-j+1} && \text{for } i=1,\ldots,n;\, j=1,\ldots,r\\
(M_2)_{ij} &= \partial^{j-1}[\exp(\xi(\boldsymbol{x},p_i))+1] && \text{for } i=1,\ldots,n;\, j=1,\ldots,n-r\\
(M_3)_{ij} &= p_i^{r-j} && \text{for } i=1,\ldots,n;\, j=1,\ldots,r-1\\
(M_4)_{ij} &= \partial^{j-1}[\exp(\xi(\boldsymbol{x},p_i))+1] && \text{for } i=1,\ldots,n;\, j=1,\ldots,n-r+1.
\end{aligned}$$

From (3.2) it is clear that we may scale τ and τ' independently by constant factors, and simultaneously by the exponential of a linear function of $x=x_1$, without changing the solution of (3.1). Thus, in the $C=0$ limit, by multiplying τ and τ' each by $\exp(-\frac{1}{2}\Sigma_{i=1}^{n}\xi(\boldsymbol{x},p_i))$ and certain constants, we find that

$$\tau = \tau_{1;0}^{-1;1}(\tfrac{1}{2}\boldsymbol{x};\tfrac{1}{2}\boldsymbol{x}) \pm \tau_{0;1}^{0;0}(\tfrac{1}{2}\boldsymbol{x};\tfrac{1}{2}\boldsymbol{x}) \qquad \tau' = \tau_{0;1}^{1;-1}(\tfrac{1}{2}\boldsymbol{x};\tfrac{1}{2}\boldsymbol{x}) \mp \tau_{1;0}^{0;0}(\tfrac{1}{2}\boldsymbol{x};\tfrac{1}{2}\boldsymbol{x})$$

where the signs depend on the parity of r. The factors of $\frac{1}{2}$ in the argument of the τ-functions correspond to those which appear as a consequence of the reduction process described in section 2.

Hence by comparison with the results of section 3, we have shown that the '$pq=0$' reduction in the single-component hierarchy exactly corresponds to the standard '$p=-q$' reduction of the two-component hierarchy.

Appendix 1

In this appendix we list the equations of the two-component hierarchy, obtained using (2.9), that are used in this paper in the reduced form (4.10)

$$\Delta_{1,0;1,0}^{-1,1;0,0} = \pm(ff^* + gg^* - FF^{(1)}) = 0 \tag{A1.1a}$$

$$\Delta_{0,1;0,0}^{0,0;-1,1} = \pm(\mathrm{D}_{x_1}^{(2)} F\cdot g + Gf^*) = 0 \tag{A1.1b}$$

$$\Delta_{0,0;0,1}^{-1,1;0,0} = \pm(\mathrm{D}_{x_1}^{(1)} F\cdot g^* + G^* f) = 0 \tag{A1.1c}$$

$$\Delta_{0,0;0,1}^{0,1;-1,0} = \pm(\mathrm{D}_{x_1}^{(2)} F\cdot f - Gg^*) = 0 \tag{A1.1d}$$

$$\Delta_{0,1;0,0}^{-1,0;0,1} = \pm(\mathrm{D}_{x_1}^{(1)} F\cdot f^* - G^* g) = 0 \tag{A1.1e}$$

$$\Delta_{0,1;1,0}^{0,1;-1,0} = \pm(\mathrm{D}_{x_1}^{(1)} f\cdot g - G^{(1)} F) = 0 \tag{A1.1f}$$

$$\Delta_{1,0;0,1}^{-1,0;0,1} = \pm(\mathrm{D}_{x_1}^{(2)} f^*\cdot g^* - G^{(1)*} F) = 0 \tag{A1.1g}$$

$$\Delta_{0,1;0,0}^{-1,1;0,0} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(1)} + \mathrm{D}_{x_1}^{(1)^2}) F\cdot g = 0 \tag{A1.2a}$$

$$\Delta_{0,0;0,1}^{0,0;-1,1} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(2)} + \mathrm{D}_{x_1}^{(2)^2}) F\cdot g^* = 0 \tag{A1.2b}$$

$$\Delta_{0,0;0,1}^{-1,2;0,-1} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(1)} + \mathrm{D}_{x_1}^{(1)2}) F\cdot f = 0 \tag{A1.2c}$$

$$\Delta_{0,1;0,0}^{0,-1;-1,2} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(2)} + \mathrm{D}_{x_1}^{(2)2}) F\cdot f^* = 0 \tag{A1.2d}$$

$$\Delta_{0,0;1,1}^{1,0;-2,1} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(2)} + \mathrm{D}_{x_1}^{(2)2}) f\cdot g^* = 0 \tag{A1.2e}$$

$$\Delta_{1,1;0,0}^{-2,1;1,0} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(1)} + \mathrm{D}_{x_1}^{(1)2}) f^*\cdot g = 0 \tag{A1.2f}$$

$$\Delta_{1,0;0,1}^{-2,2;1,-1} - \Delta_{0,1;1,0}^{-1,1;0,0} = \pm\tfrac{1}{2}(\mathrm{D}_{x_2}^{(1)} - \mathrm{D}_{x_1}^{(1)^2})(f\cdot f^* + g\cdot g^*) = 0 \tag{A1.2g}$$

$$\Delta_{0,0;0,0}^{-1,1;-1,1} = \pm(\mathrm{D}_{x_1}^{(1)}\mathrm{D}_{x_1}^{(2)} F\cdot F - 2GG^*) = 0 \tag{A1.3}$$

$$\tilde{\mathrm{D}}_{x_1}^{(1)}\Delta_{0,0;1,1}^{1,0;-2,1} = \pm\tfrac{1}{2}\mathrm{D}_{x_1}^{(1)}(\mathrm{D}_{x_2}^{(2)} + \mathrm{D}_{x_1}^{(2)^2}) f\cdot g^* = 0 \tag{A1.4a}$$

$$\tilde{\mathrm{D}}_{x_1}^{(2)}\Delta_{1,1;0,0}^{-2,1;1,0} = \pm\tfrac{1}{2}\mathrm{D}_{x_1}^{(2)}(\mathrm{D}_{x_2}^{(1)} + \mathrm{D}_{x_1}^{(1)^2}) f\cdot g^* = 0 \tag{A1.4b}$$

$$\tilde{\mathrm{D}}_{x_1}^{(2)}(\Delta_{1,0;0,1}^{-2,2;1,-1} - \Delta_{0,1;1,0}^{-1,1;0,0}) = \pm\tfrac{1}{2}\mathrm{D}_{x_1}^{(2)}(\mathrm{D}_{x_2}^{(1)} - \mathrm{D}_{x_1}^{(1)^2})(f\cdot f^* - g\cdot g^*) = 0. \tag{A1.4c}$$

Appendix 2

Here we establish (3.13):

$$\bar{v}_x + \bar{u}\bar{v} = 2\partial_x[\ln(A^+/A^{-*})].$$

By definitions (3.11) and (3.12)

$$\bar{v}_x + \bar{u}\bar{v} = \left(\frac{(A+A^{+*}-A^-A^{-*})}{FF^{(1)}}\,\frac{FF^{(1)}}{A^+A^{-*}}\right)_x \frac{A^+A^{-*}}{FF^{(1)}}$$

$$= \left(\frac{A^{+*}}{A^{-*}} - \frac{A^-}{A^+}\right)_x \frac{A^+A^{-*}}{FF^{(1)}}$$

$$= \left(\frac{\mathrm{D}_x A^{+*}\cdot A^{-*}}{A^{-*2}} + \frac{\mathrm{D}_x A^+\cdot A^-}{A^{+2}}\right)\frac{A^+A^{-*}}{FF^{(1)}}.$$

Using (3.10*d*, *e*) we get

$$\bar{v}_x + \bar{u}\bar{v} = 4\left(\frac{G^{(1)*}A^+}{A^{-*}F^{(1)}} + \frac{G^{(1)}A^{-*}}{A^+F^{(1)}}\right)$$

and by (3.10*b*, *c*), and since $F^{(1)}/F = G^{(1)}/G$, we get

$$\bar{v}_x + \bar{u}\bar{v} = 2\left(\frac{\mathrm{D}_x F\cdot A^{-*}}{FA^{-*}} - \frac{\mathrm{D}_x F\cdot A^+}{FA^+}\right)$$

$$= 2\partial_x([\ln(F/A^{-*}) - \ln(F/A^+)])$$

$$= 2\partial_x[\ln(A^+/A^{-*})]$$

as required.

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